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A Technique for the Construction of Compactly Supported Biorthogonal Wavelets of $L^2(R^n)$, $n \geq 2$

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In this paper, a technique for the concrete construction of compactly supported biorthogonal wavelet bases of $L^2(R^n)$ is given. This technique does not depend on the dimension n, and it gives rise to non-separable multidimensional wavelet bases. Of special interest is the study of the stability of the constructed wavelet frames. It is assumed that the dilation matrices associated with the wavelet bases are given by $D = 2I_n$. © 2000 Academic Press

Key Words: multidimensional wavelets; non-separable biorthogonal wavelet bases; Fourier transform; Riesz bases; stable bases.

1. INTRODUCTION

Our objective in this work is to provide the reader with a relatively easy method for the constructing n-D (n-dimensional) nonseparable biorthogonal wavelets. We shall mention that an n-D biorthogonal wavelet basis of $L^2(R^n)$ is a Riesz basis given by a set of functions $\boldsymbol{\Psi}^i_{j\mathbf{k}}$, $\tilde{\boldsymbol{\Psi}}^i_{j\mathbf{k}}$, $i = 1, 2^n - 1$, derived from $2^n - 1$ mother wavelets through the equalities

$$\boldsymbol{\Psi}^i_{j\mathbf{k}} = 2^{-jn/2}\boldsymbol{\Psi}^i(2^{-j}\boldsymbol{x} - \mathbf{k}), \qquad \tilde{\boldsymbol{\Psi}}^i_{j\mathbf{k}} = 2^{-jn/2}\tilde{\boldsymbol{\Psi}}^i(2^{-j}\boldsymbol{x} - \mathbf{k}),$$
$$j \in Z, \mathbf{k} \in Z^n. \quad (1.1)$$

Under these notations, any function $f \in L^2(R^n)$ is written in the form

$$f(\boldsymbol{x}) = \sum_{i=1}^{2^n-1} \sum_{j\in Z, \mathbf{k}\in Z^n} \langle f, \boldsymbol{\Psi}^i_{j\mathbf{k}}\rangle \tilde{\boldsymbol{\Psi}}^i_{j\mathbf{k}}(\boldsymbol{x}) = \sum_{i=1}^{2^n-1} \sum_{j\in Z, \mathbf{k}\in Z^n} \langle f, \tilde{\boldsymbol{\Psi}}^i_{j\mathbf{k}}\rangle \boldsymbol{\Psi}^i_{j\mathbf{k}}(\boldsymbol{x}). \quad (1.2)$$

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Here, the scalar product $\langle \cdot , \cdot \rangle$ is given by

$$\langle f, \mathbf{\Psi}^i_{j\mathbf{k}} \rangle = \int_{R^n} f(\boldsymbol{x})\, \mathbf{\Psi}^i_{j\mathbf{k}}(\boldsymbol{x})\, dx.$$

The construction of nonseparable and compactly supported multidimensional wavelets is a difficult problem to solve, in general [13]. However, it is well known that the design of nonseparable bidimensional or even higher dimensional biorthogonal wavelet bases is possible; see [2, 8, 9]. In [8], the authors have given an interesting and efficient method for the construction of nonseparable multidimensional and compactly supported biorthogonal wavelet bases. Their method starts with a filter of an n-D fundamental and compactly supported refinable function. Then an iterative process is applied to generate a smoother and compactly supported n-D biorthogonal wavelet basis. In this work, we prove that, for a given integer $n \geq 2$, it is possible to construct dual scaling functions $\mathbf{\Phi}(\boldsymbol{x}), \tilde{\mathbf{\Phi}}(\boldsymbol{x})$ and the $2^n - 1$ dual wavelets $\mathbf{\Psi}^i_{j\mathbf{k}}(\boldsymbol{x}), \tilde{\mathbf{\Psi}}^i_{j\mathbf{k}}(\boldsymbol{x})$, $i = 1, 2^n - 1$. The design technique is described as follows. An adapted McClellan transformation is used to provide us with two-band n-D low-pass wavelet filters $H_0(\omega_1, \omega_2, \dots, \omega_n)$, $\tilde{H}_0(\omega_1, \omega_2, \dots, \omega_n)$. Next, a choice of $n - 1$ matrices D_i, $i = 1, n - 1$ is made in such a way that if $\boldsymbol{\omega}$ denotes the point $(\omega_1, \omega_2, \dots, \omega_n) \in R^n$ and $D_0 = I_n$ is the identity matrix, then

$$\mathscr{H}_0(\boldsymbol{\omega}) = \prod_{k=0}^{n-1} H_0(D_k D_{k-1} \cdots D_1 D_0 \boldsymbol{\omega}) \qquad \text{and}$$

$$\tilde{\mathscr{H}}_0(\boldsymbol{\omega}) = \prod_{k=0}^{n-1} \tilde{H}_0(D_k D_{k-1} \cdots D_1 D_0 \boldsymbol{\omega})$$

give rise to the scaling functions $\mathbf{\Phi}(\boldsymbol{x}), \tilde{\mathbf{\Phi}}(\boldsymbol{x})$ and consequently to a biorthogonal multiresolution analysis of $L^2(R^n)$. Since the n-D filters are generated from their 1-D counterparts, we show that the main properties of the constructed multidimensional wavelets are easily deducible. Once the scaling functions have been constructed, one is faced with the problem of designing the $2^n - 1$ dual wavelets $\mathbf{\Psi}^i(\boldsymbol{x})$, $\tilde{\mathbf{\Psi}}^i(\boldsymbol{x})$. This last problem is difficult to solve in general. However, in our case it will be shown that for arbitrary dimension n one can explicitly construct these dual wavelets. Note that the wavelet filters $\mathscr{H}_0$ constructed in this work can be used by the method given in [8] and can generate other examples of n-D biorthogonal wavelet bases.

This work is organized as follows. In Section 2, we describe the technique by which one can design n-D filters or masks, candidates for generating biorthogonal scaling functions and the corresponding dual wavelet basis. In Section 3, we check the stability of the dual wavelet frames generated by the n-D dual filters of Section 2. This is done by estimating the pointwise decay of the Fourier transforms of the mother scaling functions. In Section 4, we investigate the use of the transition operator based sharper method for checking the stability of our expected n-D nonseparable dual wavelet basis. In particular, we show how to explicitly construct a finite dimensional and simple subspace which is invariant under the action of such a transition operator. Finally, in Section 5, we illustrate the results of this paper by giving some special constructions in dimensions two and three.

It will be convenient to let $\boldsymbol{\omega}$ denote the point $(\omega_1, \omega_2, \ldots, \omega_n)$, $\pi_n = (\pi, \pi, \ldots, \pi) \in R^n$, and D_0 be the identity matrix I_n. Finally, note that the wavelets to be considered in this work are supposed to be real and compactly supported.

2. DESIGN OF n-DIMENSIONAL DUAL WAVELET FILTERS

2.1. *Definitions*

In this section we provide the reader with some definitions which will be used frequently in this work.

DEFINITION 1. A matrix $D \in Z^{n \times n}$ is said to be a dilation matrix if all its singular values σ_i, $i = 1, \ldots, n$, are larger than 1.

The multiresolution analysis concept [12, 13] plays a major role in the design of wavelets. The biorthogonal version of this concept is given as follows.

DEFINITION 2. A biorthogonal multiresolution analysis is a pair of decreasing sequences of subspaces of $L^2(R^n), (V_j)_j, (\tilde{V}_j)_j$. Each sequence satisfies the following two properties:

(P_1) $\bigcup_{-\infty}^{+\infty} V_j$ is dense in $L^2(R^n)$ and $\bigcap_{-\infty}^{+\infty} V_j = \{0\}$;

(P_2) $\forall f \in L^2(R^n)$ and $\forall j \in Z$, $f(\boldsymbol{x}) \in V_{j+1} \Leftrightarrow f(2\boldsymbol{x}) \in V_j$.

Furthermore,

(P_3) there exist two scaling functions $\boldsymbol{\Phi}(\boldsymbol{x}) \in V_0$, $\tilde{\boldsymbol{\Phi}}(\boldsymbol{x}) \in \tilde{V}_0$ such that the sequences $\boldsymbol{\Phi}(\boldsymbol{x} - \mathbf{k}), \tilde{\boldsymbol{\Phi}}(\boldsymbol{x} - \mathbf{k})$, $k \in Z^n$, give rise to a biorthogonal basis of V_0.

The McClellan transformation [6] is an important ingredient in the proposed design scheme. This transformation can be defined as follows.

DEFINITION 3. Let $h(\omega) = \sum_{m=0}^{N} \alpha_m \cos(m\omega)$, $\omega, \alpha_m \in R$, be the frequency response of a 1-D zero-phase filter and let T_m be the mth degree Chebyshev polynomial. If $F(\omega_1, \omega_2, \dots, \omega_n)$ denotes a frequency response of an n-D zero phase filter, then the McClellan transformation $M_F(h)$ of $h(\omega)$ is defined by

$$M_F(h) = H_0(\boldsymbol{\omega}) = \sum_{m=0}^{N} \alpha_m T_m[F(\omega_1, \omega_2, \dots, \omega_n)].$$

Finally, the following definition will be used frequently in this paper.

DEFINITION 4. Define a set E by $E = \{0, \pi\}^n$ and let η_i be an element of E; if $\eta_j = \pi_n - \eta_i$, then η_j is said to be symmetric of η_i in E and is denoted by η_i^s. A subset A of E is said to be symmetric if $\forall \eta \in A$, $\eta^s \in A$.

2.2. *Design of n-Dimensional Low-Pass Dual Wavelet Filters*

Using properties (P_2), (P_3) of the multiresolution analysis, one concludes that the scaling function $\boldsymbol{\Phi}(\boldsymbol{x})$ has to satisfy the dilation equation

$$\boldsymbol{\Phi}(\boldsymbol{x}) = \sum_{\mathbf{k} \in Z^n} \alpha_{\mathbf{k}} \boldsymbol{\Phi}(2\boldsymbol{x} - \mathbf{k}). \tag{2.1}$$

By taking the Fourier transform from both sides of the previous equation, we conclude that

$$\hat{\boldsymbol{\Phi}}(\boldsymbol{\omega}) = \prod_{j=1}^{\infty} \mathscr{H}_0\left(\frac{\boldsymbol{\omega}}{2^j}\right), \tag{2.2}$$

where $\mathscr{H}_0(\boldsymbol{\omega}) = \sum_{\mathbf{k} \in Z^n} \alpha_{\mathbf{k}} e^{-i\mathbf{k}\cdot\boldsymbol{\omega}}$ denotes the 2^n-band low-pass wavelet filter. Consequently, to construct compactly supported n-D scaling functions, it is necessary to design finite-length low-pass filters satisfying some conditions. In our case, the first part of the design process involves the construction of a two-band n-D low-pass wavelet filter. The construction of such filters is done using McClellan transformation with the appropriate transfer function,

$$F(\omega_1, \omega_2, \dots, \omega_n) = \frac{1}{n} \sum_{i=1}^{n} \cos(\omega_i).$$

As a result, we construct two-band low-pass wavelet filters $H_0(\boldsymbol{\omega})$, $\tilde{H}_0(\boldsymbol{\omega})$, satisfying

$$H_0(\boldsymbol{\omega})\tilde{H}_0(\boldsymbol{\omega}) + H_0(\boldsymbol{\omega} + \pi_n)\tilde{H}_0(\boldsymbol{\omega} + \pi_n) = 1, \qquad \forall \boldsymbol{\omega} \in R^n. \tag{2.3}$$

More details on this construction can be found in the following proposition which is a trivial generalization to the multidimensional case of the 2-D McClellan transformation used in [4]. Note that the proof of this proposition is straightforward.

PROPOSITION 1. *Let $h(\omega), \tilde{h}(\omega)$ be 1-D wavelet filters satisfying*

$$h(\omega) = \sum_{m=0}^{N} \alpha_m \cos(m\omega) = \left(\cos\frac{\omega}{2}\right)^p P_{N-p}(\cos\omega),$$

$$\tilde{h}(\omega) = \sum_{m=0}^{N} \tilde{\alpha}_m \cos(m\omega) = \left(\cos\frac{\omega}{2}\right)^{\tilde{p}} \tilde{P}_{\tilde{N}-\tilde{p}}(\cos\omega),$$

$$h(\omega)\tilde{h}(\omega) + h(\omega+\pi)\tilde{h}(\omega+\pi) = 1, \qquad \forall\omega \in R. \tag{2.4}$$

Define $H_0(\boldsymbol{\omega}), \tilde{H}_0(\boldsymbol{\omega})$ by

$$H_0(\boldsymbol{\omega}) = \sum_{m=0}^{N} \alpha_m T_m\left[\frac{1}{n}\sum_{i=1}^{n}\cos(\omega_i)\right],$$

$$\tilde{H}_0(\boldsymbol{\omega}) = \sum_{m=0}^{\tilde{N}} \tilde{\alpha}_m T_m\left[\frac{1}{n}\sum_{i=1}^{n}\cos(\omega_i)\right].$$

Then H_0 and $\tilde{H}_0$ are written in the forms

$$H_0(\boldsymbol{\omega}) = \left(\frac{1}{n}\sum_{i=1}^{n}\cos^2\frac{\omega_i}{2}\right)^p \mathscr{F}_1(\omega_1, \dots, \omega_n),$$

$$\tilde{H}_0(\boldsymbol{\omega}) = \left(\frac{1}{n}\sum_{i=1}^{n}\cos^2\frac{\omega_i}{2}\right)^{\tilde{p}} \tilde{\mathscr{F}}_1(\omega_1, \dots, \omega_n).$$

Moreover, the couple $(H_0, \tilde{H}_0)$ is a solution of (2.3).

Notation. The scaling function associated with the two-band wavelet filter $H_0(\boldsymbol{\omega})$ will be denoted by $\phi(\cdot)$. It is defined through its Fourier transform by

$$\hat{\phi}(\boldsymbol{\omega}) = \prod_{j=1}^{\infty} H_0\left(\frac{\boldsymbol{\omega}}{2^j}\right). \tag{2.5}$$

Remark 1. It is important to mention that the number of zeros at π of the one-dimensional wavelet filter is unchanged under the McClellan transformation. As will be seen later, this property is critical in our design of smooth n-D wavelets.

It is well known [1, 7, 9, 13, 14] that in order to design dual scaling functions of Definition 2, it is necessary to construct low-pass wavelet filters $\mathscr{H}_0, \tilde{\mathscr{H}}_0$ satisfying

$$\sum_{i=0}^{2^n-1} \mathscr{H}_0(\boldsymbol{\omega} + \boldsymbol{\eta}_i)\tilde{\mathscr{H}}_0(\boldsymbol{\omega} + \boldsymbol{\eta}_i) = 1, \qquad \forall \boldsymbol{\omega} \in R^n, \tag{2.6}$$

where $\boldsymbol{\eta}_i$, $i = 0, \ldots, 2^n - 1$, are the different points of the set $E = \{0, \pi\}^n$. Note that due to the increasing number of constraints, McClellan transformation cannot possibly generate low-pass filters that satisfy (2.6). Hence, the method given in [4] cannot be used to solve Eq. (2.6). Consequently, a special technique is required in the design of $\mathscr{H}_0$ and $\tilde{\mathscr{H}}_0$. A solution to this problem is given as follows:

For $i = 1, n - 1$, construct a matrix $D_i \in Z^{n \times n}$ satisfying the following three properties:

(a) $\forall \boldsymbol{\eta}_j \in E$, $\exists \boldsymbol{\eta}_j^1 \in E$ such that $D_i(\boldsymbol{\eta}_j) = D_i(\boldsymbol{\eta}_j^1)\mathrm{mod}(2\pi Z^n)$,

(b) if $\boldsymbol{\eta}_{j'} \neq \boldsymbol{\eta}_j$ and $\boldsymbol{\eta}_{j'} \neq \boldsymbol{\eta}_j^1$, then $D_i(\boldsymbol{\eta}_j) \neq D_i(\boldsymbol{\eta}_{j'})$,

(c) $D_i D_{i-1} \cdots D_1(E)$ is a symmetric subset of E.

Define $\mathscr{H}_0, \tilde{\mathscr{H}}_0$ by

$$\mathscr{H}_0(\boldsymbol{\omega}) = \prod_{k=0}^{n-1} H_0(D_k D_{k-1} \cdots D_1 D_0 \boldsymbol{\omega}), \tag{2.7}$$

$$\tilde{\mathscr{H}}_0(\boldsymbol{\omega}) = \prod_{k=0}^{n-1} \tilde{H}_0(D_k D_{k-1} \cdots D_1 D_0 \boldsymbol{\omega}). \tag{2.7$'$}$$

Then the following proposition holds.

PROPOSITION 2. *The pair of filters* $(\mathscr{H}_0, \tilde{\mathscr{H}}_0)$ *given by* (2.7) *and* (2.7)$'$ *is a solution of* (2.6).

Proof. We first arrange the set E as follows: $\forall i = 0, 2^{n-1} - 1$, we let $\boldsymbol{\eta}_i^1 = \boldsymbol{\eta}_{2^n-1-i}$. Since

$$D_1(\boldsymbol{\eta}_i) = D_1(\boldsymbol{\eta}_i^1), \qquad \forall \boldsymbol{\eta}_i \in E,$$

one concludes that

$$
\begin{aligned}
&\sum_{i=0}^{2^n-1} \mathscr{H}_0(\boldsymbol{\omega}+\boldsymbol{\eta}_i)\tilde{\mathscr{H}}_0(\boldsymbol{\omega}+\boldsymbol{\eta}_i)\\
&\quad= \sum_{i=0}^{2^{n-1}-1} \left[H_0(\boldsymbol{\omega}+\boldsymbol{\eta}_i)\tilde{H}_0(\boldsymbol{\omega}+\boldsymbol{\eta}_i) + H_0(\boldsymbol{\omega}+\boldsymbol{\eta}_i^s)\tilde{H}_0\big(\boldsymbol{\omega}+\boldsymbol{\eta}_i^1\big)\right]\\
&\qquad \cdot \prod_{k=0}^{n-1} H_0\big(D_k D_{k-1} \cdots D_1 D_0(\boldsymbol{\omega}+\boldsymbol{\eta}_i)\big)\\
&\qquad \times \tilde{H}_0\big(D_k D_{k-1} \cdots D_1 D_0(\boldsymbol{\omega}+\boldsymbol{\eta}_i)\big)\\
&\quad= \sum_{i=0}^{2^{n-1}-1} \prod_{k=0}^{n-1} H_0\big(D_k D_{k-1} \cdots D_1 D_0(\boldsymbol{\omega}+\boldsymbol{\eta}_i)\big)\\
&\qquad \times \tilde{H}_0\big(D_k D_{k-1} \cdots D_1 D_0(\boldsymbol{\omega}+\boldsymbol{\eta}_i)\big).
\end{aligned}
$$

Using the fact that $D_2, D_3, \dots, D_{n-1}$ satisfy conditions (a), (b), (c), and repeating the above factorization technique by use of the matrices $D_2 D_1$, $D_3 D_2 D_1, \dots, D_{n-1} D_{n-2} \dots D_1$, one easily concludes that

$$
\begin{aligned}
&\sum_{i=0}^{2^n-1} \mathscr{H}_0(\boldsymbol{\omega}+\boldsymbol{\eta}_i)\tilde{\mathscr{H}}_0(\boldsymbol{\omega}+\boldsymbol{\eta}_i)\\
&\quad= H_0\big(D_{n-1} D_{n-2} \cdots D_1(\boldsymbol{\omega}+\boldsymbol{\eta}_0)\big)\tilde{H}_0\big(D_{n-1} D_{n-2} \cdots D_1(\boldsymbol{\omega}+\boldsymbol{\eta}_0)\big)\\
&\qquad + H_0\big(D_{n-1} D_{n-2} \cdots D_1\big(\boldsymbol{\omega}+\boldsymbol{\eta}_0^s\big)\big)\\
&\qquad \times \tilde{H}_0\big(D_{n-1} D_{n-2} \cdots D_1\big(\boldsymbol{\omega}+\boldsymbol{\eta}_0^s\big)\big) = 1.
\end{aligned}
$$

A concrete example of a set D_i, $i = 1, n-1$, is given as follows. For i between 1 and $n-1$, consider the matrix D_i whose action on the usual basis $(e_j)_{j=1,n}$ of R^n is given by

$$
\begin{aligned}
D_i(e_k) &= e_k, && \text{if } k \neq i, i+1,\\
D_i(e_i) &= (2k+1)e_i + (2l+1)e_{i+1} && \text{and}\\
D_i(e_{i+1}) &= 2pe_i + 2qe_{i+1}, && k, l, p, q \in Z.
\end{aligned}
$$

The set $(D_i)_i$ clearly satisfies properties a, b, and c.

In most of the applications, one needs a basis of $W_0 = V_0 \backslash V_1$, the complement of V_1 in the subspace V_0, where V_0 and V_1 are given by Definition 2. It is known that a set of $2^n - 1$ mother wavelets $\boldsymbol{\Psi}^i(\boldsymbol{x})$ is needed to construct the basis of W_0 in the orthonormal case [7]. A similar result holds in the n-D biorthogonal case. In this case, one needs to construct a set of $2^n - 1$ dual mother wavelets $\boldsymbol{\Psi}^i(\boldsymbol{x}), \tilde{\boldsymbol{\Psi}}^i(\boldsymbol{x})$. The construction of the high-pass filters that generate these dual wavelets is the subject of the next section.

2.3. *Construction of n-Dimensional High-Pass Dual Wavelet Filters*

It is well known [1, 7, 9, 13] that by using a standard technique based on Fourier transform, one concludes that to design the different mother wavelets $\boldsymbol{\Psi}^i(\boldsymbol{x}), \tilde{\boldsymbol{\Psi}}^i(\boldsymbol{x})$, $i = 1, 2^n - 1$, it is necessary to design $2^n - 1$ dual high-pass wavelet filters $\mathscr{H}_i, \tilde{\mathscr{H}}_i$. These high-pass filters together with the previously defined filters $\mathscr{H}_0, \tilde{\mathscr{H}}_0$ have to satisfy the equations

$$\sum_{i=0}^{2^n-1} \mathscr{H}_j(\boldsymbol{\omega} + \boldsymbol{\eta}_i)\overline{\tilde{\mathscr{H}}_{j'}(\boldsymbol{\omega} + \boldsymbol{\eta}_i)} = \delta_{jj'}, \tag{2.8}$$

where $\delta_{jj'} = 0$ if $j \neq j'$ and if $j = j'$. Here the $(\boldsymbol{\eta}_i)_i$ denote the different points of the set $E = \{0, \pi\}^n$. Note that Eq. (2.8) can be written in the matrix form

$$\begin{bmatrix} \overline{\tilde{\mathscr{H}}_0(\boldsymbol{\omega})} & \overline{\tilde{\mathscr{H}}_0(\boldsymbol{\omega} + \boldsymbol{\eta}_1)} & \cdots & \overline{\tilde{\mathscr{H}}_0(\boldsymbol{\omega} + \boldsymbol{\eta}_{2^n-1})} \\ \overline{\tilde{\mathscr{H}}_1(\boldsymbol{\omega})} & \overline{\tilde{\mathscr{H}}_1(\boldsymbol{\omega} + \boldsymbol{\eta}_1)} & \cdots & \overline{\tilde{\mathscr{H}}_1(\boldsymbol{\omega} - \boldsymbol{\eta}_{2^n-1})} \\ \vdots & \vdots & & \vdots \\ \overline{\tilde{\mathscr{H}}_{2^n-1}(\boldsymbol{\omega})} & \overline{\tilde{\mathscr{H}}_{2^n-1}(\boldsymbol{\omega} + \boldsymbol{\eta}_1)} & \cdots & \overline{\tilde{\mathscr{H}}_{2^n-1}(\boldsymbol{\omega} + \boldsymbol{\eta}_{2^n-1})} \end{bmatrix}$$

$$\times \begin{bmatrix} \mathscr{H}_0(\boldsymbol{\omega}) & \mathscr{H}_1(\boldsymbol{\omega}) & \cdots & \mathscr{H}_{2^n-1}(\boldsymbol{\omega}) \\ \times\mathscr{H}_0(\boldsymbol{\omega} + \boldsymbol{\eta}_1) & \mathscr{H}_1(\boldsymbol{\omega} + \boldsymbol{\eta}_1) & \cdots & \mathscr{H}_{2^n-1}(\boldsymbol{\omega} + \boldsymbol{\eta}_1) \\ \vdots & \vdots & & \vdots \\ \mathscr{H}_0(\boldsymbol{\omega} + \boldsymbol{\eta}_{2^n-1}) & \mathscr{H}_1(\boldsymbol{\omega} + \boldsymbol{\eta}_{2^n-1}) & \cdots & \mathscr{H}_{2^n-1}(\boldsymbol{\omega} + \boldsymbol{\eta}_{2^n-1}) \end{bmatrix} = I_{2^n}.$$

Under the previous notations, the different mother wavelets $\boldsymbol{\Psi}^i$, $i = 1, 2^n - 1$, are given via their Fourier transforms by

$$\hat{\boldsymbol{\Psi}}^i(2\boldsymbol{\omega}) = \mathscr{H}_i(\boldsymbol{\omega})\hat{\boldsymbol{\Phi}}(\boldsymbol{\omega}). \tag{2.9}$$

Similar formulae hold for $\hat{\Psi}^i$, $i = 1, 2^n - 1$. In general, solving Eq. (2.8) is a difficult problem. In our case, this problem is considerably simplified due to the cascade structure of the low-pass wavelet filters $\mathscr{H}_0, \tilde{\mathscr{H}}_0$. A solution to (2.8) is given by the following proposition.

PROPOSITION 3. *Let H_0 and $\tilde{H}_0$, be the wavelet filters of Proposition* 1. *Let $D_1, D_2, \ldots, D_{n-1}$ be the dilation matrices of Section* 2. *Define two filters H_1 and $\tilde{H}_1$ by*

$$H_1(\boldsymbol{\omega}) = e^{-i\omega_1}\tilde{H}_0(\boldsymbol{\omega} + \pi_n) \qquad \text{and} \qquad \tilde{H}_1(\boldsymbol{\omega}) = e^{-i\omega_1}H_0(\boldsymbol{\omega} + \pi_n).$$

If $\mathscr{H}_i, \tilde{\mathscr{H}}_i$ are the filters defined by

$$\mathscr{H}_i(\boldsymbol{\omega}) = \prod_{j=1}^{n-1}\left[\epsilon_j^i H_0(D_j D_{j-1}\ldots D_0\boldsymbol{\omega}) + \left(1 - \epsilon_j^i\right)H_1(D_j D_{j-1}\ldots D_0\boldsymbol{\omega})\right],$$

$$\tilde{\mathscr{H}}_i(\boldsymbol{\omega}) = \prod_{j=0}^{n-1}\left[\epsilon_j^i \tilde{H}_0(D_j D_{j-1}\ldots D_0\boldsymbol{\omega}) + \left(1 - \epsilon_j^i\right)\tilde{H}_1(D_j D_{j-1}\ldots D_0\boldsymbol{\omega})\right],$$

where $(\epsilon_0^i, \epsilon_1^i, \ldots, \epsilon_{n-1}^i)_{i=1,2^n-1}$ are the different points of $\{0,1\}^n \setminus (0,0,\ldots,0)$, then $\mathscr{H}_i, \tilde{\mathscr{H}}_i$, $i = 1, 2^n - 1$, is a solution of (2.8).

Proof. The proof is carried out by induction on the number of dilation matrices used in the design process. Clearly, if only the identity matrix D_0 has been used, then the family of filters H_0, H_1, $\tilde{H}_0$, and $\tilde{H}_1$ is a solution of (2.8). Assume that m dilation matrices have been used to design the set $(\mathscr{H}_i^m, \tilde{\mathscr{H}}_i^m)_{i=0,2^m-1}$, a solution of (2.8). We prove that by adding a dilation matrix D_{m+1}, the set of filters $(\mathscr{H}_i^{m+1}, \tilde{\mathscr{H}}_i^{m+1})_{i=0,2^m-1}$, given by Proposition 3, is also a solution of (2.8). We first note that by adding an extra matrix D_{m+1}, according to Proposition 3, the new filter $\mathscr{H}_i^{m+1}$ is given either by

$$\mathscr{H}_i^{m+1} = \mathscr{H}_i^m H_0(D_{m+1}D_m \cdots D_1\boldsymbol{\omega})$$

or by

$$\mathscr{H}_i^{m+1} = \mathscr{H}_i^m \tilde{H}_0(D_{m+1}D_m \cdots D_1\boldsymbol{\omega}).$$

Similar expressions hold for $\tilde{\mathscr{H}}_i^{m+1}$. Consequently,

$$\sum_{i=0}^{2^m-1} \mathscr{H}_j^{m+1}(\boldsymbol{\omega} + \boldsymbol{\eta}_i)\tilde{\mathscr{H}}_{j'}^{m+1}(\boldsymbol{\omega} + \boldsymbol{\eta}_i) + \mathscr{H}_j^{m+1}(\boldsymbol{\omega} + \boldsymbol{\eta}_i^s)\tilde{\mathscr{H}}_{j'}^{m+1}(\boldsymbol{\omega} + \boldsymbol{\eta}_i^s)$$

$$\begin{aligned}
&= \sum_{i=0}^{2^{m-1}-1} \left[\mathscr{H}_j^m(\boldsymbol{\omega} + \boldsymbol{\eta}_i)\tilde{\mathscr{H}}_{j'}^m(\boldsymbol{\eta} + \boldsymbol{\eta}_i) + \mathscr{H}_j^m(\boldsymbol{\omega} + \boldsymbol{\eta}_i^s)\tilde{\mathscr{H}}_{j'}^m(\boldsymbol{\omega} + \boldsymbol{\eta}_i^s)\right] \\
&\quad \cdot\left[\tilde{H}_0\big(D_{m+1}D_m \cdots D_1((\boldsymbol{\omega}) + \boldsymbol{\eta}_0\big) \right.\\
&\quad \left. \times H_0\big(D_{m+1}D_m \cdots D_1((\boldsymbol{\omega}) - \boldsymbol{\eta}_0\big)\right] \\
&\quad + \sum_{i=2^{m-1}}^{2^m-1} \left[\mathscr{H}_j^m(\boldsymbol{\omega} + \boldsymbol{\eta}_i)\tilde{\mathscr{H}}_{j'}^m(\boldsymbol{\omega} + \boldsymbol{\eta}_i)\right.\\
&\quad \left. + \mathscr{H}_j^m(\boldsymbol{\omega} + \boldsymbol{\eta}_i^s)\tilde{\mathscr{H}}_{j'}^m(\boldsymbol{\omega} + \boldsymbol{\eta}_i^s)\right] \\
&\quad \cdot\left[\tilde{H}_0\big(D_{m+1}D_m \cdots D_1((\boldsymbol{\omega}) + \boldsymbol{\eta}_0^s\big) \right.\\
&\quad \left. \times H_0\big(D_{m+1}D_m \cdots D_1((\boldsymbol{\omega}) + \boldsymbol{\eta}_0^s\big)\right].
\end{aligned}$$

Using the induction hypothesis, the first terms of the previous two sums are equal to $\delta_{jj'}$. Hence, the induction hypothesis holds for $m + 1$. Consequently, the set of wavelet filters given by Proposition 3 is a solution of (2.8).

Remark 2. Once a high-pass wavelet filter

$$\mathscr{H}_i(\boldsymbol{\omega}) = \sum_{\mathbf{k}} \alpha_{\mathbf{k}}^i e^{i\mathbf{k}\cdot\boldsymbol{\omega}}$$

has been constructed, the corresponding wavelet $\tilde{\boldsymbol{\Psi}}^i(\boldsymbol{x})$ is given by

$$\boldsymbol{\Psi}^i(\boldsymbol{x}) = \sum_{\mathbf{k}} \alpha_{\mathbf{k}}^i \boldsymbol{\Phi}(2\boldsymbol{x} - \mathbf{k}).$$

Similarly, we construct the dual wavelets $\tilde{\boldsymbol{\Psi}}^i(\boldsymbol{x})$, $i = 1, 2^n - 1$.

Remark 3. The reader can check that high-pass filters corresponding to a biorthogonal multiresolution analysis with dilation matrix $D_i D_{i-1} \cdots D_1$ are easily obtained by a trivial modification of Proposition 3. Hence, our construction can also be used to generate nonseparable n-D biorthogonal wavelet bases with dilation matrix $M_i = D_i D_{i-1} \cdots D_1$, where i is any integer satisfying $1 \leq i \leq n - 1$.

3. POINTWISE DECAY OF $\hat{\Phi}$ AND STABILITY

It is important to mention that solving Eq. (2.8) does not ensure the construction of a biorthogonal wavelet basis of $L^2(R^n)$. In fact, the constructed mother wavelets have to generate a Riesz basis or equivalently a stable basis of $L^2(R^n)$. In the special case of a biorthogonal wavelet basis, it is shown in [3] that $\boldsymbol{\Psi}_{j\mathbf{k}}^i, \tilde{\Psi}_{j\mathbf{k}}^i$ is a Riesz basis of biorthogonal wavelets if there exists $C > 0$ such that $\forall f \in L^2(R^n)$,

$$\sum_{i=1}^{2^n-1} \sum_{j\in Z, \mathbf{k}\in Z^n} \left|\langle f, \boldsymbol{\Psi}_{j\mathbf{k}}^i \rangle\right|^2 \leq C\|f\|_2^2, \tag{3.1}$$

$$\sum_{i=1}^{2^n-1} \sum_{j\in Z, \mathbf{k}\in Z^n} \left|\langle f, \tilde{\boldsymbol{\Psi}}_{j\mathbf{k}}^i \rangle\right|^2 \leq C\|f\|_2^2. \tag{3.1$'$}$$

In [2, 3], the authors have shown that the above conditions are satisfied if the translates of each dual scaling functions $\boldsymbol{\Phi}, \tilde{\boldsymbol{\Phi}}$ are Riesz bases for the subspace that they generate. Note that $\boldsymbol{\Phi}(\cdot - \mathbf{k})$, $\mathbf{k} \in Z^n$, is a Riesz (or stable) basis if there exist two positive constants $A \geq B > 0$ [10] such that

$$B \leq \sum_{k\in Z^n} \left|\hat{\boldsymbol{\Phi}}(\boldsymbol{\omega} - \mathbf{k})\right|^2 \leq A, \qquad \text{a.e. in } R^n. \tag{3.2}$$

The stability of the translates of general refinable functions has been extensively studied by many authors [3, 10, 11, 14].

In this section, we give the first method that ensures the stability of the translates $\boldsymbol{\Phi}(\cdot - \mathbf{k})$, $\mathbf{k} \in Z^n$, $\tilde{\boldsymbol{\Phi}}(\cdot - \mathbf{k})$, $\mathbf{k} \in Z^n$, where $\boldsymbol{\Phi}, \tilde{\boldsymbol{\Phi}}$ are generated from the n-D low-pass wavelet filters $\mathscr{H}_0$ and $\tilde{\mathscr{H}}_0$ of the previous section. This method is based on the pointwise decay of the Fourier transform of the dual scaling functions. By a generalization of Theorem 3.2 in [5], it is shown in [9] that if the Fourier transforms of $\boldsymbol{\Phi}(\boldsymbol{x})$ and $\tilde{\boldsymbol{\Phi}}(\boldsymbol{x})$ satisfy the inequalities

$$\left|\hat{\boldsymbol{\Phi}}(\boldsymbol{\omega})\right| \leq c\left(1 + \|\boldsymbol{\omega}\|_2^2\right)^{-\epsilon - n/4}, \qquad \left|\hat{\tilde{\boldsymbol{\Phi}}}(\boldsymbol{\omega})\right| \leq \tilde{c}\left(1 + \|\boldsymbol{\omega}\|_2^2\right)^{-\tilde{\epsilon} - n/4}, \tag{3.3}$$

for some constants $c, \tilde{c}$ and real numbers $\epsilon, \tilde{\epsilon} > 0$, then $\boldsymbol{\Phi}(\boldsymbol{x}), \tilde{\boldsymbol{\Phi}}(\boldsymbol{x})$ give rise to a stable biorthogonal wavelet basis of $L^2(R^n)$ with dilation matrix $2I_n$. Based on this observation, we are required to estimate the decay of $\hat{\boldsymbol{\Phi}}(\boldsymbol{\omega})$ and $\hat{\tilde{\boldsymbol{\Phi}}}(\boldsymbol{\omega})$. This estimate is to be done under the weak condition that there exists $n_0 \in \mathbf{N}\backslash\{0\}$ such that

$$\sup_{2\mathbf{k}n_0\pi \leq \boldsymbol{\omega} \leq 2(\mathbf{k}+\mathbf{1})n_0\pi} \left|\hat{\phi}(\boldsymbol{\omega})\right| \leq \sup_{2\mathbf{k}'n_0\pi \leq \boldsymbol{\omega} \leq 2(\mathbf{k}'+\mathbf{1})n_0\pi} \left|\hat{\phi}(\boldsymbol{\omega})\right|, \tag{3.4}$$

where $\hat{\phi}(\boldsymbol{\omega})$ is given by (2.4) and $\mathbf{k}, \mathbf{k}'$ are arbitrary points in $\mathbf{N}^n$ satisfying $\mathbf{k} \geq \mathbf{k}' \geq \mathbf{0}$. Here the order "$\geq$" is defined on R^n by $\boldsymbol{x} = (x_1, \ldots, x_n)$; $\mathbf{y} = (y_1, \ldots, y_n) \in R^n$; and $\boldsymbol{x} \geq \mathbf{y}$ means that $x_i \geq y_i$, $\forall i = 1, n$. Note that the condition (3.4) will be used in the proof of Theorem 1.

Next, we define a subset $\mathbf{S}$ of R^n by

$$\boldsymbol{\omega} \in \mathbf{S} \Leftrightarrow \boldsymbol{\omega} = \left(\epsilon_1 2^{k_1}, \ldots, \epsilon_n 2^{k_n}\right)\pi + \left(\alpha_1, \ldots, \alpha_n\right),$$

where $\forall i = 1, n$, $\alpha_i \in [0, 2n_p \pi]$ and $\epsilon_i \in \{0, 1\}$. Under the above notation, we prove the following lemma which plays an important role in the decay estimate of $\hat{\boldsymbol{\Phi}}(\boldsymbol{\omega})$.

LEMMA 1. *Let* $\boldsymbol{\omega}$ *be an arbitrary point in* $\mathbf{S}$; *then there exists a fixed constant* C *satisfying*

$$\prod_{j=1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^{n} \cos^2 \frac{\omega_i}{2^j} \right) \leq \frac{C}{1 + \|\boldsymbol{\omega}\|_2^2}. \tag{3.5}$$

Proof. We first remark that

$$\begin{aligned}
&\frac{1}{n} \sum_{i=1}^{n} \cos^2 \frac{\omega_i}{2^j} \\
&\quad = 1 - \frac{1}{n} \sum_{i=1}^{n} \sin^2 \frac{\omega_i}{2^j} \\
&\quad \leq \frac{1}{\sum_{i=1}^{n} \sin^2 \frac{\omega_i}{2^j}} \left[\sum_{i=1}^{n} \sin^2 \frac{\omega_i}{2^j} \left(1 - \sin^2 \frac{\omega_i}{2^j}\right) + \sum_{i=1}^{n} \frac{n-1}{n} \sin^4 \frac{\omega_i}{2^j} \right] \\
&\quad \leq \left[\frac{\sum_{i=1}^{n} \sin^2 \frac{\omega_i}{2^j} \left(1 - \sin^2 \frac{\omega_i}{2^j}\right)}{\sum_{i=1}^{n} \sin^2 \frac{\omega_i}{2^j}} \right] \left[1 + \frac{\sum_{i=1}^{n} \frac{n-1}{n} \sin^4 \frac{\omega_i}{2^j}}{\sum_{i=1}^{n} \sin^2 \frac{\omega_i}{2^j} - \sum_{i=1}^{n} \sin^4 \frac{\omega_i}{2^j}} \right].
\end{aligned} \tag{3.6}$$

We prove that the infinite product of the second term corresponding to the right-hand side of (3.6) is bounded by a fixed constant c. To this end, we first prove that there exists a fixed number of j's such that for an arbitrary point $\boldsymbol{\omega} \in \mathbf{S}$, $\sin^2(\omega/2^j)$ is larger than $\frac{1}{2}$. In other words, if we define the subset $F(\boldsymbol{\omega})$ by

$$F(\boldsymbol{\omega}) = \left\{ j \in \mathbf{N}; \sin^2 \frac{\omega}{2^j} \geq \frac{1}{2}, i = 1, \ldots, n \right\}, \tag{3.7}$$

then $\forall \boldsymbol{\omega} \in \mathbf{S}$, $|F(\boldsymbol{\omega})|$ is bounded by a fixed constant $m \in \mathbf{N}$. Without loss of generality, we may assume that $\omega_i \geq 0$, then $\boldsymbol{\omega} \in \mathbf{S}$ implies that there exists $k_i \geq 0$ such that $\omega_i = \epsilon_i 2^{k_i}\pi + \alpha_i$, $\alpha_i \in [0, 2n_0\pi]$. If N_1, N_2 are the integers given by

$$N_1 = \left[2 + \frac{\log(n_0\pi)}{\log 2}\right] + 1, \qquad N_2 = \left[1 + \frac{\log((2n_0+1)\pi)}{\log 2}\right] + 1,$$

where $[x]$ denotes the integer part of x, then some elementary computations show that if $k_i \geq N_1$ we have

$$\sin^2\frac{\omega_i}{2^j} \leq \left(\frac{\alpha_i}{2^j}\right)^2 < \frac{1}{2}, \qquad \forall k_i \geq j \geq N_1,$$

$$\sin^2\frac{\omega_i}{2^j} \leq \left(\pi 2^{k_i - j} + \frac{\alpha_i}{2^j}\right)^2 < \frac{1}{2}, \qquad \forall j \geq k_i + N_2.$$

Moreover, if $k_i < N_1$, then

$$\sin^2\frac{\omega_i}{2^j} \leq \left(\pi 2^{k_i - j} + \frac{\alpha_i}{2^j}\right)^2 < \frac{1}{2}, \qquad \forall j \geq N_1 + N_2.$$

Consequently, there are at most $N_1 + N_2 - 1$ values of js for which $\sin^2(\omega_i/2^j)$ can be larger than $\frac{1}{2}$. Hence $|F(\boldsymbol{\omega})| \leq n(N_1 + N_2 - 1)$ and we have

$$\sin^2\left(\frac{\omega_i}{2^j}\right) < \frac{1}{2}, \qquad \forall i = 1, n \text{ and } j \notin F(\boldsymbol{\omega}).$$

Thus

$$0 \leq \frac{\sin^4(\omega_i/2^j)}{\sin^2(\omega_i/2^j) - \sin^4(\omega_i/2^j)} < 1, \qquad \forall i = 1, n \text{ and } j \notin F(\boldsymbol{\omega}).$$

It is well known that if $(x_j)_j$ is an infinite sequence of real numbers such that for all $j \geq 1$, $x_j \geq 1$, then to prove that $\prod_{j=1}^{\infty} x_j$ is bounded it is enough to prove that $\sum_{j=1}^{\infty} \log(x_j)$ is also bounded. Consequently, to prove that

$$\prod_{j=1}^{\infty}\left[1 + \left(\sum_{i=1}^{n} \frac{n-1}{n}\sin^4\frac{\omega_i}{2^j}\right) \Big/ \left(\sum_{i=1}^{n}\sin^2\frac{\omega_i}{2^j} - \sum_{i=1}^{n}\sin^4\frac{\omega_i}{2^j}\right)\right]$$

is bounded, we have only to prove that

$$\sum_{j=1}^{\infty} \log\left(1 + \left(\sum_{i=1}^{n} \frac{n-1}{n} \sin^4 \frac{\omega_i}{2^j}\right) \Bigg/ \left(\sum_{i=1}^{n} \sin^2 \frac{\omega_i}{2^j} - \sum_{i=1}^{n} \sin^4 \frac{\omega_i}{2^j}\right)\right)$$

is also bounded. Since $\sin^2(\omega_i/2^j) < \frac{1}{2}$, $j \notin F(\boldsymbol{\omega})$, then

$$\begin{aligned}
&\sum_{j=1}^{\infty} \log\left(1 + \left(\sum_{i=1}^{n} \frac{n-1}{n} \sin^4 \frac{\omega_i}{2^j}\right) \Bigg/ \left(\sum_{i=1}^{n} \sin^2 \frac{\omega_i}{2^j} - \sum_{i=1}^{n} \sin^4 \frac{\omega_i}{2^j}\right)\right) \\
&\quad \leq \sum_{j=1}^{\infty} \left(\sum_{i=1}^{n} \frac{n-1}{n} \sin^4 \frac{\omega_i}{2^j}\right) \Bigg/ \left(\sum_{i=1}^{n} \sin^2 \frac{\omega_i}{2^j} - \sum_{i=1}^{n} \sin^4 \frac{\omega_i}{2^j}\right) \\
&\quad \leq \sum_{j=1}^{\infty} 2\left(\sum_{i=1}^{n} \sin^4 \frac{\omega_i}{2^j}\right) \Bigg/ \left(\sum_{i=1}^{n} \sin^2 \frac{\omega_i}{2^j}\right) \\
&\quad \leq 2 \sum_{j=1}^{\infty} \left(\sum_{i=1}^{n} \sin^2 \frac{\omega_i}{2^j}\right).
\end{aligned}$$

Consequently, $\forall \boldsymbol{\omega} \in \mathbf{S}$, we have

$$\begin{aligned}
&\sum_{j=1}^{\infty} \log\left(1 + \left(\sum_{i=1}^{n} \frac{n-1}{n} \sin^4 \frac{\omega_i}{2^j}\right) \Bigg/ \left(\sum_{i=1}^{n} \sin^2 \frac{\omega_i}{2^j} - \sum_{i=1}^{n} \sin^4 \frac{\omega_i}{2^j}\right)\right) \\
&\quad \leq 2 \sum_{j=1}^{\infty} \left(\sum_{i=1}^{n} \sin^2 \frac{\omega_i}{2^j}\right) + n(N_1 + N_2 - 1).
\end{aligned}$$

At this point, we need to show that for an arbitrary $\boldsymbol{\omega} \in \mathbf{S}$, the inequality

$$\sum_{j=1}^{\infty} \sin^2 \frac{\omega_i}{2^j} \leq c$$

holds for a fixed constant c. We may assume that $\omega_i = 2^{k_i}\pi + \alpha_i$, $\alpha_i \in [0, 2n_0\pi]$, $k_i \geq N_1$. The case where $\omega_i = \alpha_i$ is treated in a similar way. We are implicitly assuming that $k_i \geq N_1$, if $k_i < N_1$; then the obvious changes are left to the reader. By writing the infinite sum $\sum_{j=1}^{\infty} \sin^2(\omega_i/2^j)$ into four different sums,

$$\begin{aligned}
\sum_{j=1}^{\infty} \sin^2 \frac{\omega_i}{2^j} = \sum_{j=1}^{N_1 - 1} \sin^2 \frac{\omega_i}{2^j} + \sum_{j=N_1}^{k_i} \sin^2 \frac{\omega_i}{2^j} + \sum_{j=k_i+1}^{k_i+N_2-1} \sin^2 \frac{\omega_i}{2^j} \\
+ \sum_{j=k_i+N_2}^{\infty} \sin^2 \frac{\omega_i}{2^j},
\end{aligned}$$

one concludes that

$$\begin{aligned}\sum_{j=1}^{\infty} \sin^2 \frac{\omega_i}{2^j} &\le (N_1 - 1) + \sum_{j=N_1}^{k_i} \left(\frac{\alpha_i}{2^j} \right) + (N_2 - 1) \\ &\quad + \sum_{j=k_i+N_2}^{\infty} \left(\pi 2^{k_i - j} + \frac{\alpha_i}{2^j} \right)^2 \\ &< (N_1 + N_2 - 2) + \sum_{j=1}^{\infty} \left(\frac{\alpha_i}{2^j} \right)^2 + \sum_{j=N_2}^{\infty} \left(\frac{\pi}{2^j} + \frac{\alpha_i}{2^j} \right)^2 \\ &< (N_1 + N_2 - 2) + \frac{\pi^2}{3} \left(4n_0^2 + \frac{4n_0^2 + 4n_0 + 1}{4^{N_2 - 1}} \right) = c.\end{aligned}$$

Note that the constant c appearing in the above inequality does not depend on the given ω_i, thus

$$2 \sum_{j=1}^{\infty} \sum_{i=1}^{n} \sin^2 \frac{\omega_i}{2^j} < 2nc.$$

Consequently,

$$\begin{aligned}&\prod_{j=1}^{\infty} \left[1 + \left(\sum_{i=1}^{n} \frac{n-1}{n} \sin^4 \frac{\omega_i}{2^j} \right) \Big/ \left(\sum_{i=1}^{n} \sin^2 \frac{\omega_i}{2^j} - \sum_{i=1}^{n} \sin^4 \frac{\omega_i}{2^j} \right) \right] \\ &\qquad \le e^{n(2c + N_1 + N_2 - 1)} = C_1.\end{aligned}$$

It remains to evaluate the infinite product of the left term corresponding to the right-hand side of (3.6). This is easily done as follows. Let $k \ge 1$ be a positive integer, then

$$\begin{aligned}&\prod_{j=1}^{k} \left(\sum_{i=1}^{n} \sin^2 \frac{\omega_i}{2^j} \left(1 - \sin^2 \frac{\omega_i}{2^j} \right) \Big/ \sum_{i=1}^{n} \sin^2 \frac{\omega_i}{2^j} \right) \\ &\quad = \prod_{j=1}^{k} \left(\sum_{i=1}^{n} \sin^2 \frac{\omega_i}{2^{j-1}} \Big/ 4 \sum_{i=1}^{n} \sin^2 \frac{\omega_i}{2^j} \right) \\ &\quad = \frac{\sum_{i=1}^{n} \sin^2 \omega_i}{\sum_{i=1}^{n} \omega_i^2 \left(2^k / \omega_i \right)^2 \sin^2 \left(\omega_i / 2^k \right)}.\end{aligned} \tag{3.8}$$

As $k \to +\infty$, $(2^k/\omega_i)^2 \sin^2(\omega_i/2^k) \to 1$, hence

$$\prod_{j=1}^{\infty}\left[\left(\sum_{i=1}^{n} \sin^2\frac{\omega_i}{2^j}\left(1-\sin^2\frac{\omega_i}{2^j}\right)\right)\Big/\left(\sum_{i=1}^{n}\sin^2\frac{\omega_i}{2^j}\right)\right]$$
$$= \frac{\sum_{i=1}^{n}\sin^2\omega_i}{\sum_{i=1}^{n}\omega_i^2} \le \frac{C_2}{1+\|\boldsymbol{\omega}\|_2^2}. \tag{3.9}$$

Collecting everything together, we conclude that $\forall\boldsymbol{\omega}\in\mathbf{S}$, we have

$$\prod_{j=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\cos^2\frac{\omega_i}{2^j}\right) \le \frac{C_1C_2}{1+\|\boldsymbol{\omega}\|_2^2}.$$

Once the above lemma has been established and under hypothesis (3.4), the following theorem provides us with an easy way to estimate the decay of $\hat{\boldsymbol{\Phi}}(\boldsymbol{\omega})$.

THEOREM 1. *Let* $h(\omega) = \sum_{m=0}^{N}\alpha_m\cos(m\omega) = (\cos(\omega/2))^{2p}P(\cos\omega)$ *be a low-pass wavelet filter. Define an n-D low-pass wavelet filter* $H_0(\boldsymbol{\omega})$ *by* $H_0(\boldsymbol{\omega}) = \sum_{m=0}^{N}\alpha_m T_m[(1/n)\sum_{i=1}^{n}\cos\omega_i]$. *Let* $k\ge 1$ *be a positive integer and define a real number* β *by*

$$\beta = \sup_{\boldsymbol{\omega}\in R^n}\left|\prod_{j=1}^{k-1}P\left(\frac{1}{n}\sum_{i=1}^{n}\cos(2^j\omega_i)\right)\right|^{1/k}. \tag{3.10}$$

Then

$$\hat{\boldsymbol{\Phi}}(\boldsymbol{\omega}) = \prod_{j=1}^{\infty}\prod_{k=0}^{n-1}H_0\left(D_kD_{k-1}\cdots D_0\frac{\boldsymbol{\omega}}{2^j}\right)$$

satisfies

$$\left|\hat{\boldsymbol{\Phi}}(\boldsymbol{\omega})\right| \le C\left(1+\|\boldsymbol{\omega}\|_2^2\right)^{-n(p-(\log\beta)/(2\log 2))},$$

for some constant C.

Proof. We first note that since $\hat{\boldsymbol{\Phi}}(\boldsymbol{\omega})$ is symmetric, then it is enough to prove its decay in the region $\boldsymbol{\omega} > \mathbf{0}$. Note that if $Q(\boldsymbol{\omega}) = \sum_0^N\alpha_m((1/n)\sum_{i=1}^{n}\cos\omega_i)^m$ with $Q(0)=1$, then $Q(\omega)$ is differentiable on R^n; moreover, $\|Q(\boldsymbol{\omega})\|_2 \le 1 + C\|\boldsymbol{\omega}\|_2$, $\forall\boldsymbol{\omega}\in R^n$. In particular, if $\|\boldsymbol{\omega}\|_2\le 1$, then

$$\prod_{j=1}^{\infty}Q\left(\frac{\boldsymbol{\omega}}{2^j}\right) \le \prod_{j=1}^{\infty}e^{c\|\boldsymbol{\omega}\|_2/2^j} \le C.$$

To estimate the decay of $\hat{\boldsymbol{\Phi}}(\boldsymbol{\omega})$, we first estimate the decay of the infinite product $\prod_{j=1}^{\infty} H_0(\frac{\boldsymbol{\omega}}{2^j})$, where $\boldsymbol{\omega}$ is an arbitrary point of the set $\mathbf{S}$. By using a result from Proposition 1, one concludes that

$$H_0(\boldsymbol{\omega}) = \left(\frac{1}{n}\sum_{i=1}^{n}\cos^2\frac{\omega_i}{2}\right)^p P\left(\frac{1}{n}\sum_{i=1}^{n}\cos\,\omega_i\right).$$

Define the function $\phi_1(\boldsymbol{\omega})$ by

$$\phi_1(\boldsymbol{\omega}) = \prod_{j=1}^{\infty} P\left(\frac{1}{n}\sum_{i=1}^{n}\cos\frac{\omega_i}{2^j}\right)$$

and consider the auxiliary function $G(\boldsymbol{\omega})$ defined by

$$G(\boldsymbol{\omega}) = \prod_{j=0}^{k-1} P\left(\frac{1}{n}\sum_{i=1}^{n}\cos(2^j\omega_i)\right).$$

Let $\boldsymbol{\omega}$ be an arbitrary point of R^n; then there exist two positive integers k, l such that

$$2^{kl} \le \|\boldsymbol{\omega}\|_2 \le 2^{k(l+1)}.$$

By writing $\phi_1(\boldsymbol{\omega})$ in the form

$$\phi_1(\boldsymbol{\omega}) = \prod_{j=0}^{l} G\left(\frac{2^{-k}\boldsymbol{\omega}}{2^j}\right)\prod_{j=l+1}^{\infty} G\left(\frac{2^{-k}\boldsymbol{\omega}}{2^j}\right)$$

and using the fact that $\forall k \ge l+1$, $\|2^{-k}\boldsymbol{\omega}/2^j\|_2 \le 1$, one concludes that

$$|\phi_1(\boldsymbol{\omega})| \le c_1\left|\prod_{j=1}^{l} G\left(\frac{2^{-k}\boldsymbol{\omega}}{2^j}\right)\right|.$$

Since

$$\sup_{\boldsymbol{\omega}\in R^n}\left|\prod_{j=1}^{k-1} P\left(\frac{1}{n}\sum_{i=1}^{n}\cos(2^j\omega_i)\right)\right|^{1/k} \le \beta,$$

then

$$\left|\prod_{j=0}^{l} G\left(\frac{2^{-k}\boldsymbol{\omega}}{2^j}\right)\right| \le \beta^{k(l+1)} \le \beta^k\beta^{\log\|\boldsymbol{\omega}\|_2/\log 2}$$

$$\le c_2\left(1+\|\boldsymbol{\omega}\|_2^2\right)^{\log\,\beta/2\log 2}.$$

Consequently,

$$\left|\phi_1(\boldsymbol{\omega})\right| \leq c_1 c_2\left(1+\|\boldsymbol{\omega}\|_2^2\right)^{\log\beta/2\log 2}.$$

Using the result of Lemma 1, one concludes that

$$\left|\prod_{j=1}^{\infty} H_0\left(\frac{\boldsymbol{\omega}}{2^j}\right)\right| \leq \prod_{j=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\cos^2\frac{\omega_i}{2^j}\right)^p \left|\phi_1(\boldsymbol{\omega})\right|$$
$$\leq C\left(1+\|\boldsymbol{\omega}\|_2^2\right)^{-p+\log\beta/2\log 2}.$$

The extension of the above result to R^n is done as follows. Let $\boldsymbol{\omega}$ be an arbitrary positive point of R^n; then there exists $\mathbf{k}=(k_1,\ldots,k_n)\in\mathbf{N}^n$ and $\boldsymbol{\epsilon}=(\epsilon_1,\ldots,\epsilon_n)$, $\epsilon_i\in\{0,1\}$, such that $\boldsymbol{\omega}^1\leq\boldsymbol{\omega}\leq 2\boldsymbol{\omega}^1$ for some

$$\boldsymbol{\omega}^1\in\mathbf{S}_{\mathbf{k}}=\left(\epsilon_1 2^{k_1},\ldots,\epsilon_n 2^{k_n}\right)\pi+\left[0,2n_0\pi\right]^n\subset\mathbf{S}.$$

By using inequality (3.4) as many times as is necessary, one concludes that

$$\left|\widehat{\phi_1}(\boldsymbol{\omega})\right| \leq \sup_{\boldsymbol{\omega}'\in\mathbf{S}_{\mathbf{k}}}\left|\widehat{\phi_1}(\boldsymbol{\omega}')\right|$$
$$\leq \sup_{\boldsymbol{\omega}'\in\mathbf{S}_{\mathbf{k}}} c\left(1+\|\boldsymbol{\omega}'\|_2^2\right)^{-p+(\log\beta/2\log 2)}$$
$$\leq c'\left(1+\|\boldsymbol{\omega}'\|_2^2\right)^{-p+(\log\beta/2\log 2)}.$$

Finally, if in the above inequality we substitute $\boldsymbol{\omega}$ by $D_i D_{i-1}\cdots D_1\boldsymbol{\omega}$, $i=1n$, one can easily conclude that

$$\left|\prod_{j=1}^{\infty} H_0\left(D_i D_{i-1}\cdots D_0\frac{\boldsymbol{\omega}}{2^j}\right)\right|$$
$$\leq c'\left(1+\|D_i D_{i-1}\cdots D_0\boldsymbol{\omega}\|_2^2\right)^{-p+\log\beta/(2\log 2)}$$
$$\leq c'\left(1+\|D_i D_{i-1}\cdots D_0\|_2^2\|\boldsymbol{\omega}\|_2^2\right)^{-p+\log\beta/(2\log 2)}$$
$$\leq c_i(1+\|\boldsymbol{\omega}\|_2)^{-p+\log\beta/(2\log 2)}.$$

Consequently, there exists a constant C such that

$$\left|\hat{\boldsymbol{\Phi}}(\boldsymbol{\omega})\right| = \left|\prod_{i=0}^{n-1}\prod_{j=1}^{\infty} H_0\left(D_i D_{i-1}\cdots D_0\frac{\boldsymbol{\omega}}{2^j}\right)\right|$$
$$\leq C(1+\|\boldsymbol{\omega}\|_2)^{n(-p+\log\beta/2\log 2)}. \tag{3.11}$$

Remark 4. Some techniques of the previous proof have been used previously in the 1-D case in [5].

Remark 5. If $\hat{\mathbf{\Phi}}$ satisfies the inequality (3.11), then $\mathbf{\Phi}$ is r-Hölder continuous, where r is any real number satisfying $r < 2n(p - \frac{\log \beta}{2 \log 2} - \frac{1}{2})$.

Remark 6. Equation (3.10) can be easily checked numerically; consequently, Theorem 1 provides us with a way to estimate the decay of $\hat{\mathbf{\Phi}}$ and $\hat{\tilde{\mathbf{\Phi}}}$. This method is particularly useful when the supports of the n-D wavelet filters are large. In the case where the n-D wavelet filters have small supports, the method of the next section gives us better results.

4. ANOTHER SUFFICIENT CONDITION FOR STABILITY

It is well known that the transition operator based method is a sharp method for checking the stability of the translates of a given refinable function. The 1-D version of this method has been given in [3] and the n-D version has been given in [10]. Furthermore, in [14], the author has extended this method to the general case of refinable function vectors. In [3], the authors have given a necessary and sufficient condition that ensures the biorthogonality and the Riesz basis property of the 1-D compactly supported dual wavelets generated by the dual wavelet filters $h_0(\xi)$ and $\tilde{h}_0(\xi)$. This condition is related to the transition operators T_{h_0} and $T_{\tilde{h}_0}$ defined respectively by

$$T_{h_0}(f) = \left|h_0\left(\frac{\omega}{2}\right)\right|^2 f\left(\frac{\omega}{2}\right) + \left|h_0\left(\frac{\omega}{2} + \pi\right)\right|^2 f\left(\frac{\omega}{2} + \pi\right), \tag{4.1}$$

$$T_{\tilde{h}_0}(f) = \left|\tilde{h}_0\left(\frac{\omega}{2}\right)\right|^2 f\left(\frac{\omega}{2}\right) + \left|\tilde{h}_0\left(\frac{\omega}{2} + \pi\right)\right|^2 f\left(\frac{\omega}{2} + \pi\right). \tag{4.2}$$

It is shown that if

$$E = \left\{ \sum_{n=N_1}^{N_2} c_n e^{in\xi} \right\}, \qquad \tilde{E} = \left\{ \sum_{n=\tilde{N}_1}^{\tilde{N}_2} \tilde{c}_n e^{in\xi} \right\}$$

are two stable sets under the action of T_{h_0} and $T_{\tilde{h}_0}$ then $h_0, \tilde{h}_0$ give rise to a stable (or Riesz) basis of biorthogonal wavelets if and only if 1 is a simple eigenvalue of the restriction on E and $\tilde{E}$ of T_{h_0} and $T_{\tilde{h}_0}$, respectively. All of the other eigenvalues have to be inside the unit circle. In [10], the authors have shown that this last result is still valid in the n-D situation.

We should mention that unlike the 1-D case, the construction of invariant subspaces E and $\tilde{E}$ is no longer straightforward in the n-D case, $n \geq 2$ [4, 10]. It turns out that the n-D filters $\mathscr{H}_0$ and $\tilde{\mathscr{H}_0}$ of Section 2 have the advantage of providing us with explicit and simple invariant subspaces of the associated n-D transition operators $T_{\mathscr{H}_0}$ and $T_{\tilde{\mathscr{H}}_0}$. The following theorem gives us an invariant subspace of $T_{\mathscr{H}_0}$.

THEOREM 2. *Let* $\mathscr{H}_0(\boldsymbol{\omega}) = \sum_{m_1,\dots,m_n=-N}^{N} \alpha_{m_1,\dots,m_n} \cos[\sum_{i=1}^{n} m_i \omega_i]$ *be the n-D filter given by* (2.6). *Define the transition operator* $T_{\mathscr{H}_0}$ *by*

$$T_{\mathscr{H}_0}(f)(\boldsymbol{\omega}) = \sum_{i=1}^{2^n-1} \left|\mathscr{H}_0\left(\frac{\boldsymbol{\omega}}{2} + \boldsymbol{\eta}_i\right)\right|^2 f\left(\frac{\boldsymbol{\omega}}{2} + \boldsymbol{\eta}_i\right),$$

where $\boldsymbol{\eta}_i = (\eta_i^1, \dots, \eta_i^n)$, $i = 0, 2^n - 1$, *are the different points of the set* $\{0, \pi\}^n$. *Under the above notation, the set*

$$\mathscr{E}_N = \operatorname{span}\left\{\cos\left[\sum_{i=1}^{n} \mu_i \omega_i\right], \mu_i \in \{-2N, \dots, 2N\}\right\}$$

is an invariant set of $T_{\mathscr{H}_0}$.

Proof. Let $f(\boldsymbol{\omega}) = \cos[\sum_{i=1}^{n} \mu_i \omega_i]$, for some $\mu_i \in \{-2N, \dots, 2N\}$, $i = 1, n$. Since

$$\left|\mathscr{H}_0\left(\frac{\boldsymbol{\omega}}{2}\right)\right|^2 = \sum_{m_1,\dots,m_n=-2N}^{2N} \beta_{m_1,\dots,m_n} \cos\left[\sum_{i=1}^{n} m_i \frac{\omega_i}{2}\right],$$

then

$$\begin{aligned} T_{\mathscr{H}_0}(f)(\boldsymbol{\omega}) = \sum_{j=0}^{2^n-1} \sum_{m_1,\dots,m_n=-2N}^{2N} & \beta_{m_1,\dots,m_n} \\ & \times \cos\left[\sum_{i=1}^{n} m_i\left(\frac{\omega_i}{2} + \eta_j^i\right)\right] \cos\left[\sum_{i=1}^{n} \mu_i\left(\frac{\omega_i}{2} + \eta_j^i\right)\right]. \end{aligned}$$

Hence to prove that $T_{\mathscr{H}_0}(f)(\boldsymbol{\omega}) \in \mathscr{E}_N$, it suffices to show that for a given $(m_1, \dots, m_n) \in \{-2N, \dots, 2N\}^n$, the sum

$$J(\boldsymbol{\omega}) = \sum_{j=0}^{2^n-1} \cos\left[\sum_{i=1}^{n} m_i\left(\frac{\omega_i}{2} + \eta_j^i\right)\right] \cos\left[\sum_{i=1}^{n} \mu_i\left(\frac{\omega_i}{2} + \eta_j^i\right)\right]$$

belongs to $\mathscr{E}_n$. To this end, we consider two cases:

First Case. We assume that $\forall i = 1, n$, $m_i - \mu_i \in 2Z$. Then it is clear that

$$J(\boldsymbol{\omega}) = \frac{1}{2} \sum_{j=0}^{2^n - 1} \left(\cos\left[\sum_{i=1}^{n} \left(\frac{m_i + \mu_i}{2} \right) \left(\omega_i + 2\eta_j^i \right) \right] \right.$$
$$\left. + \cos\left[\sum_{i=1}^{n} \left(\frac{m_i - \mu_i}{2} \right) \left(\omega_i + 2\eta_j^i \right) \right] \right)$$
$$= \frac{1}{2} \sum_{j=0}^{2^n - 1} \left(\cos\left[\sum_{i=1}^{n} \left(\frac{m_i + \mu_i}{2} \right) \omega_i \right] + \cos\left[\sum_{i=1}^{n} \left(\frac{m_i - \mu_i}{2} \right) \omega_i \right] \right) \in \mathscr{E}_N.$$

Second Case. We assume that there exists $i_0 \in \{1, 2, \dots, n\}$ such that

$$m_{i_0} - \mu_{i_0} \notin 2Z.$$

Without loss of generality, we may assume that m_{i_0} is even and μ_{i_0} is odd. Define two sets S_+ and S_- by

$$S_+ = \left\{ \boldsymbol{\eta}_j,\ j = 0, 2^n - 1 \text{ such that} \right.$$
$$\left. \cos\left[\sum_{i=1}^{n} m_i \left(\frac{\omega_i}{2} + \eta_j^i \right) \right] = \cos\left[\sum_{i=1}^{n} m_i \frac{\omega_i}{2} \right] \right\},$$

$$S_- = \left\{ \boldsymbol{\eta}_j,\ j = 0, 2^n - 1 \text{ such that} \right.$$
$$\left. \cos\left[\sum_{i=1}^{n} m_i \left(\frac{\omega_i}{2} + \eta_j^i \right) \right] = -\cos\left[\sum_{i=1}^{n} m_i \frac{\omega_i}{2} \right] \right\}.$$

Note that since m_{i_0} is even, then S_+ is written as $S_+ = S_+^1 \cup S_+^2$, where

$$S_+^1 = \left\{ \boldsymbol{\eta}_j \in S_+;\ \eta_j^{i_0} = 0 \right\}, \qquad S_+^2 = \left\{ \boldsymbol{\eta}_j \in S_+;\ \eta_j^{i_0} = \pi \right\}.$$

Similarly, we write $S_- = S_-^1 \cup S_-2$. Since m_{i_0} is even, then $|S_+^1| = |S_+^2|$. It is easy to see that

$$\forall \boldsymbol{\eta}_{1j} = \left(\eta_{1j}^1, \dots, \eta_{1j}^n \right) \in S_+^1, \qquad \exists \boldsymbol{\eta}_{2j} = \left(\eta_{2j}^1, \dots, \eta_{2j}^n \right) \in S_+^2$$
$$\text{such that } \begin{cases} \eta_{1j}^i = \eta_{2j}^i, \ \forall i \neq i_0 \\ \eta_{1j}^{i_0} = 0,\ \eta_{2j}^{i_0} = \pi. \end{cases}$$

Moreover, since μ_{i_0} is odd, then

$$\sum_{\boldsymbol{\eta}_j \in S_+^2} \cos\left[\sum_{i=1}^{n} \mu_i\left(\frac{\omega_i}{2} + \eta_j^i\right)\right] = -\sum_{\boldsymbol{\eta}_j \in S_+^1} \cos\left[\sum_{i=1}^{n} \mu_i\left(\frac{\omega_i}{2} + \eta_j^i\right)\right].$$

It follows that

$$\begin{aligned} J_1(\boldsymbol{\omega}) &= \sum_{\boldsymbol{\eta}_j \in S_+} \cos\left[\sum_{i=1}^{n} m_i\left(\frac{\omega_i}{2} + \eta_j^i\right)\right]\cos\left[\sum_{i=1}^{n} \mu_i\left(\frac{\omega_i}{2} + \eta_j^i\right)\right] \\ &= \cos\left[\sum_{i=1}^{n} m_i\left(\frac{\omega_i}{2}\right)\right] \sum_{\boldsymbol{\eta}_j \in S_+^1 \cup S_+^2} \cos\left[\sum_{i=1}^{n} \mu_i\left(\frac{\omega_i}{2} + \eta_j^i\right)\right] \\ &= \cos\left[\sum_{i=1}^{n} m_i\left(\frac{\omega_i}{2}\right)\right] \sum_{\boldsymbol{\eta}_j \in S_+^1} \left(\cos\left[\sum_{i=1}^{n} \mu_i\left(\frac{\omega_i}{2} + \eta_j^i\right)\right]\right. \\ &\qquad \left. - \cos\left[\sum_{i=1}^{n} \mu_i\left(\frac{\omega_i}{2} + \eta_j^i\right)\right]\right) = 0. \end{aligned}$$

Similarly, one concludes that

$$J_2(\boldsymbol{\omega}) = \sum_{\boldsymbol{\eta}_j \in S_-} \cos\left[\sum_{i=1}^{n} m_i\left(\frac{\omega_i}{2} + \eta_j^i\right)\right]\cos\left[\sum_{i=1}^{n} \mu_i\left(\frac{\omega_i}{2} + \eta_j^i\right)\right] = 0.$$

Consequently, $J(\boldsymbol{\omega}) = J_1(\boldsymbol{\omega}) + J_2(\boldsymbol{\omega}) = 0 \in \mathscr{E}_N$.

Collecting everything together, one concludes that for arbitrary $(m_1, \ldots, m_n) \in \{-2N, \ldots, 2N\}^n$ and $f(\boldsymbol{\omega}) \in \mathscr{E}_N$,

$$J(\boldsymbol{\omega}) = \sum_{j=0}^{2^n - 1} \cos\left[\sum_{i=1}^{n} m_i\left(\frac{\omega_i}{2} + \eta_j^i\right)\right] f\left(\frac{\omega_i}{2} + \eta_j^i\right) \in \mathscr{E}_N.$$

Hence, $T_{\mathscr{H}_0}(f)(\boldsymbol{\omega}) \in \mathscr{E}_N$ and consequently $\mathscr{E}_N$ is invariant under the action of $T_{\mathscr{H}_0}$.

The following corollary shows that the invariant subspace $\mathscr{E}_N$ given by the above theorem can be reduced to a smaller invariant subspace.

COROLLARY 1. *Let* $\mathscr{H}_0^2(\boldsymbol{\omega}) = \sum_{m_1, \ldots, m_n = -2N}^{2N} \beta_{m_1, \ldots, m_n} \cos[\sum_{i=1}^{n} m_i \omega_i]$ *and let* $\mathscr{E}_N^1$ *be the subspace of* $\mathscr{E}_N$ *defined by*

$$\mathscr{E}_n^1 = \operatorname{span}\left\{\cos\left[\sum_{i=1}^{n} k_i \omega_i\right] \textit{ such that } \mathbf{k} = (k_1, \ldots, k_n) \in \operatorname{supp}\mathscr{H}_0^2\right\}.$$

Then $\mathscr{E}_N^1$ *is stable under the action of* $T_{\mathscr{H}_0}$.

Proof. Since $\forall \mathbf{k} = (k_1, \ldots, k_n) \notin [-2N, 2N]^n$, we have $\beta_{\mathbf{k}} = 0$, then $\mathbf{k} \notin \operatorname{supp} \mathscr{H}_0^2$. Consequently, $\mathscr{E}_N^1$ is a subspace of $\mathscr{E}_N$. Moreover, since $\operatorname{supp} \mathscr{H}_0^2$ is symmetric with respect to the origin, then for all $\mathbf{k}, \mathbf{m} \in \operatorname{supp} \mathscr{H}_0^2$, with $\mathbf{k} - \mathbf{m} \in 2Z^n$, we have $(\mathbf{k} + \mathbf{m})/2 \in \operatorname{supp} \mathscr{H}_0^2$ and $(\mathbf{k} + \mathbf{m})/2 \in$
$\operatorname{supp} \mathscr{H}_0^2$. Hence, by using the two cases given in the proof of the previous theorem, one concludes that $\mathscr{E}_N^1$ is stable under the action of $T_{\mathscr{H}_0}$.

Finally, the following corollary provides us with an explicit formula for computing the matrix corresponding to the restriction on $\mathscr{E}_N^1$ of $T_{\mathscr{H}_0}$.

COROLLARY 2. *Let* $\mathscr{H}_0^2(\boldsymbol{\omega}) = \sum_{m_1, \ldots, m_n = -2N}^{2N} \beta_{m_1, \ldots, m_n} \cos[\sum_{i=1}^n m_i \omega_i]$ *and let* $U_{\mathbf{k}} = \cos[\sum_{i=1}^n k_i \omega_i]$ *be an arbitrary vector of the basis of* $\mathscr{E}_N^1$. *Then*

$$T_{\mathscr{H}_0} U_{\mathbf{k}} = 2^{n-1} \sum_{\mathbf{m} \in \operatorname{supp} \mathscr{H}_0^2} \prod_{i=1}^{n} \gamma_{m_i - k_i} \beta_{m_1, \ldots, m_n} J_{\mathbf{mk}},$$

where

$$J_{\mathbf{mk}} = \cos\left[\sum_{i=1}^{n} \left(\frac{m_i + k_i}{2}\right) \omega_i\right] + \cos\left[\sum_{i=1}^{n} \left(\frac{m_i - k_i}{2}\right) \omega_i\right]$$

and $\gamma_{m_i - k_i} = 1$ *if* $m_i - k_i = 0 \bmod(2)$, *and* 0 *otherwise*.

Proof. The proof follows from the two cases given in the proof of the previous theorem.

As we have already mentioned, if 1 is a simple eigenvalue of $T_{\mathscr{H}_0}$ and $T_{\tilde{\mathscr{H}}_0}$, restricted respectively to $\mathscr{E}_N^1$ and $\tilde{\mathscr{E}}_N^1$, and all the other eigenvalues lie inside the unit circle, then the filters $\mathscr{H}_0$ and $\tilde{\mathscr{H}}_0$ of Section 2 generate a n-D biorthogonal wavelet basis. Finally, it is shown in [14] that the mother scaling function $\boldsymbol{\Phi}$ and the corresponding mother wavelets $\boldsymbol{\Psi}^i$, $i = 1, 2^n - 1$, are r-Hölder continuous $\forall r < -\log|\lambda_{\mathscr{E}_N}|/(2 \log 2) - n/2$, where $\lambda_{\mathscr{E}_N}$ is the largest eigenvalue different from 1 of $T_{\mathscr{H}_0}$.

5. EXAMPLES

In this section, we provide the reader with three constructions that illustrate the techniques and the results of this paper. The first two examples deal with the construction of biorthogonal wavelet bases of $L^2(R^2)$. The third shows how to apply our proposed scheme for the construction of a family of biorthogonal wavelet bases of $L^2(R^3)$.

FIRST EXAMPLE. In this example, we give a trivial construction of the 2-D biorthogonal wavelet basis by applying the techniques of Section 2. Let $h_0(\omega), \tilde{h}_0(\omega)$ be 1-D wavelet filters. We assume that the translates of the associated dual scaling functions $\phi, \tilde{\phi}$ are stable. Let $H_0(\omega_1, \omega_2)$, $\tilde{H}_0(\omega_1, \omega_2)$ be the 2-D filters given by

$$H_0(\omega_1, \omega_2) = h_0(\omega_1), \qquad \tilde{H}_0(\omega_1, \omega_2) = \tilde{h}_0(\omega_1).$$

It is clear that the couple $(H_0, \tilde{H}_0)$ satisfies Eq. (2.3). Next, consider the matrix $D_1 = \left[\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right]$. It is clear that D_1 satisfies the conditions (a)–(c) of Section 2.2. Hence, by using Proposition 2, one concludes that the 2-D wavelet filters $\mathscr{H}_0, \tilde{\mathscr{H}}_0$, given by

$$\mathscr{H}_0 = H_0(w_1, w_2) H_0(D_1(\omega_1, \omega_2)) = h_0(\omega_1) h_0(\omega_1 - \omega_2),$$

$$\tilde{\mathscr{H}}_0 = \tilde{h}_0(\omega_1) \tilde{h}_0(\omega_1 - \omega_2),$$

satisfy the equation (2.6). Note that if $\mathbf{\Phi}$ denotes the scaling function generated by $\mathscr{H}_0$, then we have

$$\hat{\mathbf{\Phi}}(\omega_1, \omega_2) = \hat{\phi}(\omega_1) \hat{\phi}(\omega_1 - \omega_2).$$

Since

$$\begin{aligned} &\sum_{\alpha, \beta \in Z} \left| \hat{\mathbf{\Phi}}(\omega_1 + 2\pi\alpha, \omega_2 + \pi\beta) \right|^2 \\ &\quad = \sum_{\alpha, \beta \in Z} \left| \hat{\phi}(\omega_1 + 2\pi\alpha) \right|^2 \left| \hat{\phi}(\omega_1 - \omega_2 + 2\pi(\alpha - \beta)) \right|^2 \\ &\quad = \sum_{\alpha \in Z} \left| \hat{\phi}(\omega_1 + 2\pi\alpha) \right|^2 \cdot \sum_{\gamma \in Z} \left| \hat{\phi}(\omega_1 - \omega_2 + 2\pi\gamma) \right|^2 \end{aligned}$$

and since

$$c_1 \le \sum_{\alpha \in Z} \left| \hat{\phi}(\omega_1 + 2\pi\alpha) \right|^2 \le c_2 \qquad \text{a.e. for some } 0 < c_1 \le c_2 < \infty,$$

then

$$c_1^2 \le \sum_{\alpha, \beta \in Z} \left| \hat{\mathbf{\Phi}}(\omega_1 + 2\pi\alpha, \omega_2 + \pi\beta) \right|^2 \le c_2^2, \qquad \text{a.e.}$$

Consequently, the translates of $\mathbf{\Phi}$ form a stable family. Similarly, one can prove the same property for $\tilde{\mathbf{\Phi}}$.

SECOND EXAMPLE. This second construction starts by considering 2-D wavelet filters $H(\omega_1, \omega_2), \tilde{H}(\omega_1, \omega_2)$ satisfying Eq. (2.3). Moreover, we assume that $H, \tilde{H}$ generate a 2-D biorthogonal wavelet basis with the dilation matrix D_1 of the previous example. This type of filter has been designed in [4, 9]. The scaling functions associated with these filters are given by

$$\hat{\phi}(\omega_1, \omega_2) = \prod_{j=1}^{\infty} H\left(D_1^{-j}(\omega_1, \omega_2)\right),$$

$$\hat{\tilde{\phi}}(\omega_1, \omega_2) = \prod_{j=1}^{\infty} \tilde{H}\left(D_1^{-j}(\omega_1, \omega_2)\right).$$

Moreover, we assume that the translates of ϕ and $\tilde{\phi}$ are stable. Proposition 2 tells us that the wavelet filters

$$\mathscr{H}_0(\omega_1, \omega_2) = H(\omega_1, \omega_2) H\left(D_1(\omega_1, \omega_2)\right),$$

$$\tilde{\mathscr{H}}_0(\omega_1, \omega_2) = \tilde{H}(\omega_1, \omega_2) \tilde{H}\left(D_1(\omega_1, \omega_2)\right)$$

satisfy Eq. (2.6). Moreover, if $\mathbf{\Phi}$ is the scaling function generated by $\mathscr{H}_0$, then we have

$$\hat{\mathbf{\Phi}}(\omega_1, \omega_2) = \prod_{j=1}^{\infty} \mathscr{H}_0\left(\frac{\omega_1}{2^j}, \frac{\omega_2}{2^j}\right) = \prod_{j=1}^{\infty} H\left(\frac{\omega_1}{2^j}, \frac{\omega_2}{2^j}\right) H\left(D\left(\frac{\omega_1}{2^j}, \frac{\omega_2}{2^j}\right)\right)$$

$$= \prod_{j=1}^{\infty} H\left(D^{-2j}(\omega_1, \omega_2)\right) H\left(D^{-2j+1}(\omega_1, \omega_2)\right) = \hat{\phi}(\omega_1, \omega_2).$$

Hence the translates of $\mathbf{\Phi}$ form a stable family. Similarly, we prove that this last property is also satisfied by $\tilde{\mathbf{\Phi}}$.

Note that the techniques used in this second example are not restricted to the use of the starting dual filters $H, \tilde{H}$ with the condition that the corresponding scaling functions are stable. Moreover, the matrix D_1 can be substituted by any other matrix that satisfies conditions (a)–(c) of Section 2.2. However, in this general situation, the stability of the translates of $\mathbf{\Phi}$ and $\tilde{\mathbf{\Phi}}$ has to be tested by one of the methods given in Sections 3 and 4.

THIRD EXAMPLE. In this last example, we show how to use the proposed scheme for the construction of the biorthogonal wavelet basis of $L^2(R^3)$. We first use Proposition 1 and generate 3-D dual filters $H_0(\omega_1, \omega_2, \omega_3), \tilde{H}_0(\omega_1, \omega_2, \omega_3)$ satisfying Eq. (2.3). Since $n = 3$, we need to construct two matrices D_1 and D_2 satisfying the three conditions (a), (b), and

(c) of Section 2.2. A possible choice of these matrices is given by

$$D_1 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad D_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 0 \end{bmatrix}.$$

By using Proposition 2, one concludes that the 3-D filters given by

$$\mathscr{H}_0(\boldsymbol{\omega}) = H_0(\omega_1, \omega_2, \omega_3) H_0(D_1(\omega_1, \omega_2, \omega_3)) H_0(D_2 D_1(\omega_1, \omega_2, \omega_3)),$$
$$\tilde{\mathscr{H}}_0(\boldsymbol{\omega}) = \tilde{H}_0(\omega_1, \omega_2, \omega_3) \tilde{H}_0(D_1(\omega_1, \omega_2, \omega_3)) \tilde{H}_0(D_2 D_1(\omega_1, \omega_2, \omega_3))$$

satisfy Eq. (2.6). Finally, the stability of the translates of the resulting scaling functions has to be tested by one of the methods given in Sections 3 and 4.

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